CHAPTER 1

DERIVATIVES AND LIMITS

1.1 Introduction to the Derivative

PREREQUISITES

1. Recall how to use functional notation (Section R.6).
2. Recall how to graph straight lines and how to compute the slopes of the lines (Section R.4).

PREREQUISITE QUIZ

1. Let $f(x) = x^2 + 3x + 4$. Evaluate the following expressions:
   (a) $f(3)$
   (b) $f(y)$
   (c) $f(x + y)$
2. Find the slopes of the lines given by the following equations:
   (a) $2x + 2y = 5$
   (b) $y = -x + 3$
3. Sketch the graphs of the equations given in Question 2.

GOALS

1. Be able to calculate the velocity of an object from its position formula.
2. Be able to compute the slope of the tangent line for a given curve.

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3. Be able to reproduce and understand the formula

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad .$$

STUDY HINTS

1. Definitions. Average velocity is the ratio of distance travelled to time elapsed over a period of time. Instantaneous velocity is the same ratio as the time period approaches zero. In more general situations, average velocity corresponds to average rate of change and instantaneous velocity corresponds to the instantaneous rate of change.
2. Definition of derivative. Know that $\Delta y/\Delta x$ for Δx approaching zero is the derivative. Also, notice that $\Delta y = f(x + \Delta x) - f(x)$.
3. Cancelling Δx . If, in working out $\Delta y/\Delta x$, Δx can not be cancelled from the denominator, you should suspect a computational error in most cases. Be sure you know how to use the definition to find the derivative.
4. Physical interpretation. Instantaneous velocity and slope of the tangent line are some of the many names that the derivative specializes to in specific situations.
5. Notation. Familiarize yourself with $f'(x)$.
6. Quadratic function rule. Memorize the formula for now. A more general formula will be memorized in Section 1.4 which will subsume this one. Note what happens if $a = 0$ or if a and b are both 0 ; the linear and constant function rules result.

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SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By definition, $\Delta y = f(x_0 + \Delta x) - f(x_0)$ and the average velocity is $\Delta y/\Delta x$.

   (a) $\Delta y = f(2 + 0.5) - f(2) = [(2.5)^2 + 3(2.5)] - [(2)^2 + 3(2)] = 13.75 - 10 = 3.75$; $\Delta y/\Delta x = 3.75/0.5 = 7.5$.

   (b) $\Delta y = f(2 + 0.01) - f(2) = [(2.01)^2 + 3(2.01)] - [(2)^2 + 3(2)] = 10.0701 - 10 = 0.0701$; $\Delta y/\Delta x = 0.0701/0.01 = 7.01$.

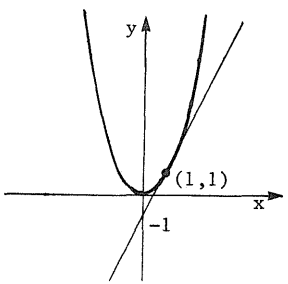
   (c) $\Delta y = f(4 + 0.1) - f(4) = [(4.1)^2 + 3(4.1)] - [(4)^2 + 3(4)] = 29.11 - 28 = 1.11$; $\Delta y/\Delta x = 1.11/0.1 = 11.1$.

   (d) $\Delta y = f(4 + 0.01) - f(4) = [(4.01)^2 + 3(4.01)] - [(4)^2 + 3(4)] = 28.1101 - 28 = 0.1101$; $\Delta y/\Delta x = 0.1101/0.01 = 11.01$.

5. The instantaneous velocity is given by $f'(x_0)$. By the quadratic function rule, $f'(x) = 2x + 3$; therefore, $f'(2) = 7$ m/sec.

9. (a) The instantaneous velocity is given by $f'(x_0)$. By the quadratic function rule, $f'(x_0) = 2x_0 + 3$.

   (b) The instantaneous velocity is 10 m/sec when $2x_0 + 3 = 10$, i.e., $x_0 = 7/2$ seconds.

13. 

The slope of the tangent line at x_0 is $f'(x_0)$. We begin by finding $\Delta y/\Delta x = [(1 + \Delta x)^2 - (1)^2]/\Delta x = [2\Delta x + (\Delta x)^2]/\Delta x = 2 + \Delta x$. As Δx gets small, we get $f'(1) = 2$.

17. By definition, $f'(x_0) = \Delta y/\Delta x$ for a small Δx . Thus, $f'(x_0) = [f(x_0 + \Delta x) - f(x_0)]/\Delta x = [(a\Delta x + 2) - 2]/\Delta x = a\Delta x/\Delta x = a$.

21. The quadratic function rule states that the derivative of $ax^2 + bx + c$ is $2ax + b$. Here, $a = 1$, $b = 1$, and $c = -1$, so the derivative is $2x + 1$. At $x_0 = 1$, $f'(1) = 3$.

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25. (a) The vertex of the parabola is the point where the derivative is zero. By the quadratic function rule, $f'(x) = 2x - 16$, which is zero at $x_0 = 8$. Substitution gives $y = (8)^2 - 16(8) + 2 = -62$, so the desired point is $(8,-62)$.

(b) Algebraically, the vertex occurs at the highest or lowest point, which may be found by completing the square. $y = (x^2 - 16x + 64) + (2 - 64) = (x - 8)^2 - 62$. $(x - 8)^2$ is zero for $x = 8$ and positive otherwise, so the parabola has its lowest point at $x = 8$.

29. By the quadratic function rule, the derivative of $ax^2 + bx + c$ is $2ax + b$. Here, $a = 1$, $b = 3$, and $c = -1$, so $f'(x) = 2x + 3$.

33. By the quadratic function rule, the derivative of $at^2 + bt + c$ is $2at + b$. Here, $a = -4$, $b = 3$, and $c = 6$, so $g'(t) = -8t + 3$.

37. As in Example 8, the best time to make the jump is when the velocities are equal. We want to equate the derivatives: $4t + 3 = 6t + 1$. Therefore, the jump should be made at $t = 1$.

41. Here, $f'(x) = 2x$, so $f'(3) = 6$.

45. Expanding, $k(y) = y^2 - 3y - 28$, so $k'(y) = 2y - 3$ and $k'(-1) = -5$.

49. $\Delta y/\Delta x = \{[4(x + \Delta x)^2 + 3(x + \Delta x) + 2] - [4x^2 + 3x + 2]\}/\Delta x = (8x\Delta x + 4(\Delta x)^2 + 3\Delta x)/\Delta x = 8x + 3 + 4\Delta x$. As Δx gets small, the derivative becomes $8x + 3$.

53. $\Delta y/\Delta x = \{[-2(x + \Delta x)^2 + 5(x + \Delta x)] - [-2x^2 + 5x]\}/\Delta x = \{-4x\Delta x - 2(\Delta x)^2 + 5\Delta x\}/\Delta x = -4x - 2\Delta x + 5$. As Δx gets small, the derivative becomes $-4x + 5$

57. From basic geometry, $A(x) = x^2$ and the perimeter $P(x) = 4x$. $A'(x) = 2x$, which is $(1/2)P(x)$.

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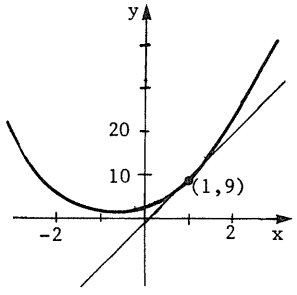
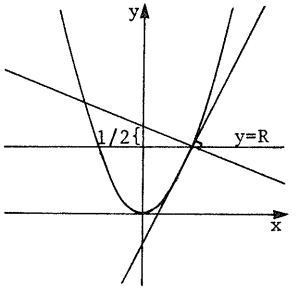
61.



By the quadratic function rule, $f'(x) = 6x + 4$, so $f'(1) = 10$. Since $f(1) = 9$, the point-slope form of the line yields $y = 9 + 10(x - 1)$ or $y = 10x - 1$. By Example 7, the vertex of a parabola occurs where $f'(x_0) = 0$. In this case $6x + 4 = 0$ at $x = -2/3$.

65.



If $f(x) = x^2$, $f'(x) = 2x$, so the slope of the tangent line is $2R$; therefore, the perpendicular line is $y = R^2 + \left(-\frac{1}{2R}\right)(x - R)$. This line crosses $x = 0$ at $y = R^2 + 1/2$. The horizontal crosses at $y = R^2$, so the distance is $1/2$.

69. (a) Let $f(t) = 24.5t - 4.9t^2$. The velocity at time t is $f'(t)$. By the quadratic function rule, $f'(t) = 2(-4.9)t + 24.5$, i.e., $f'(t) = -9.8t + 24.5$. Substituting different values of t, we have the velocity at $t = 0,1,2,3,4,5$ equal to 24.5, 14.7, 4.9, -4.9, -14.7, -24.5 meters/sec, respectively.

(b) $f(t) = 24.5t - 4.9t^2 = -4.9(t - 2.5)^2 + 30.625$ is a parabola facing downward. The highest point is at the vertex when $t = 2.5$ sec.

(c) The velocity is equal to zero when $-9.8t + 24.5 = 0$, i.e., $t = 2.5$ sec. This result is consistent with part (b).

(d) The ball strikes the ground when $f(t) = 0$. Solving for $24.5t - 4.9t^2 = 0$ gives $t = 5$ sec.

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SECTION QUIZ

1. If a particle's position is given by $y = x^2 + 5x - 7$ after x seconds, compute the particle's velocity at $y_0 = -1$. (Hint: there are two answers.)

2. Compute $f'(x)$:

   (a) $f(x) = x^2 + x + 1$

   (b) $f(x) = 3\pi^2$

3. Compute $\Delta y/\Delta x$ for $y = x^2/4 + 3x/2 + 2.7$.

4. Find the slope of the tangent line of the parabola in Question 3 at $x_0 = 4$.

5. Just a little while ago, the notorious Claude Crum ran off with sweet Cindy Lou, the southern belle daughter of Deputy Dan. Claude's position is $5x + 3$ miles after x hours. Deputy Dan's horse, who takes superduper vitamins, can run faster, so its position is $3x^2 + 2x$ at time x hours.

   (a) Find the speed of both individuals at any time x .

   (b) When will Deputy Dan catch up with Claude?

   (c) What is the speed of both individuals when Dan catches Claude and stops sweet Cindy Lou from marrying Claude?*

*Dear Reader: I realize that many of you hate math but are forced to complete this course for graduation. Thus, I have attempted to maintain interest with "entertaining" word problems. They are not meant to be insulting to your intelligence. Obviously, most of the situations will never happen; however, calculus has several practical uses and such examples are found throughout Marsden and Weinstein's text. I would appreciate your comments on whether my "unusual" word problems should be kept for the next edition.

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ANSWERS TO PREREQUISITE QUIZ

1. (a) 22

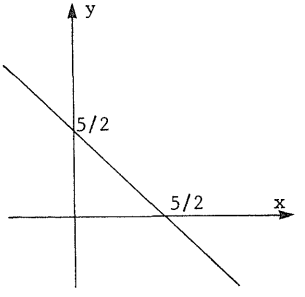
   (b) $y^2 + 3y + 4$

   (c) $(x + y)^2 + 3(x + y) + 4$

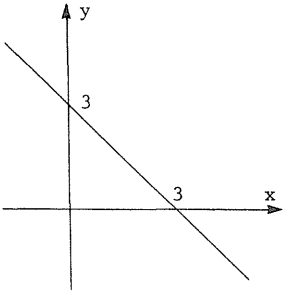
2. (a) -1

   (b) -1

3. (a)



(b)



ANSWERS TO SECTION QUIZ

1. $y'(1) = 7$ and $y'(-6) = -7$

2. (a) $2x + 1$

   (b) 0

3. $x/2 + 3/2 + \Delta x/4$

4. 7/2

5. (a) Claude: 5miles/hour; Dan: $(6x + 2)$miles/hour

   (b) $[(1 + \sqrt{5})/2]$ hours

   (c) Claude: 5miles/hour; Dan: $(5 + 3\sqrt{5})$miles/hour

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1.2 Limits

PREREQUISITES

1. Recall how to factor simple polynomials. (Section R.1)
2. Recall that graphs may be used to depict functions. (Section R.6)

PREREQUISITE QUIZ

1. Factor the following:
   (a) $x^4 - y^4$
   (b) $x^3 - 8$
   (c) $4x^2 + 16x + 16$
   (d) $3x^2 + 8x + 4$
2. Sketch the graphs of the following functions:
   (a) $y = -x + 2$
   (b) $y = x^3 - 1$

GOALS

1. Be able to find limits by computation and by reading graphs.

STUDY HINTS

1. Cautions about limits. $\lim_{x \to a} f(x)$ depends on what happens around $x = a$, not what happens at $x = a$. Also, the limiting value must be approached from both sides of $x = a$, not just one side. Example 2 demonstrates these points well.

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2. Basic properties of limits. Know how to use these statements for computing limits. Most of them are intuitively obvious, so there is no need to memorize. You should take a few moments to convince yourself that these statements are correct.

3. Replacement rule. This rule merely states that limits of equivalent functions are equal. The method of Example 3(b) is commonly used on test questions. The idea is to cancel common factors to leave a nonzero denominator. Usually, one of the factors is $(x - a)$ if we want to find the limit as x approaches a (See Example 5(a)).

4. Continuity defined. This is a very important concept in mathematics. Notice that $\lim_{x \to x_0} f(x)$ must exist and that it must equal $f(x_0)$ if f is continuous at x_0.

5. Rational functions. These functions, polynomials or quotients of polynomials, are continuous at x_0 if the denominator does not equal zero. Their limits may be computed at x_0 by "substituting" $x = x_0$; however, be sure the function is rational before applying this rule.

6. Limits of infinity. Example 8(a) shows the important fact that $\lim_{x \to \pm\infty} (1/x) = 0$. To compute limits at infinity, divide numerator and denominator by an appropriate power of x so that all except for one term in the denominator approaches zero. See Example 8, parts (b) and (c).

7. Terminology. "Limit does not exist" can either mean the limit is approaching infinity or that no particular value is being approached as x approaches x_0.

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8. One-sided limit. Notice that in Example 7, no limit exists at zero. However, as we approach zero from either the left or the right, the function does approach a certain value. This is the concept of one-sided limits. More details will come later in this course.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $f(x) = (x^3 - 3x^2 + 5x - 3)/(x - 1)$. We find $f(1.1) = 2.01$, $f(1.001) = 2.000001$, $f(0.9) = 2.01$, $f(0.999) = 2.000001$. These results suggest that $\lim_{x\to 1} f(x) = 2$. To verify this, we divide the numerator by $x - 1$, obtaining $f(x) = [(x - 1)(x^2 - 2x + 3)]/(x - 1)$ which, for $x \neq 1$, is equal to $x^2 - 2x + 3$. Applying the replacement rule and the continuity of rational functions, we have $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^2 - 2x + 3) = 1^2 - 2 \cdot 1 + 3 = 2$.

5. Using the method of Example 3, we have $\lim_{x\to 3} (17 + x) = \lim_{x\to 3} 17 + \lim_{x\to 3} x$ (sum rule) $= 17 + 3$ (constant function and identity function rules) $= 20$.

9. Using the method of Example 4, we have $\lim_{x\to 3} [(x^2 - 9)/(x^2 + 3)] = \lim_{x\to 3} (x^2 - 9)/\lim_{x\to 3} (x^2 + 3)$ (quotient rule) $= \left[\lim_{x\to 3} x^2 + \lim_{x\to 3} (-9)\right] / \left[\lim_{x\to 3} x^2 + \lim_{x\to 3} 3\right]$ (sum rule) $= (9 - 9)/(9 + 3)$ (power and constant function rule) $= 0$.

13. First, we note that $(u - \sqrt{3})/(u^2 - 3) = (u - \sqrt{3})/(u - \sqrt{3})(u + \sqrt{3}) = 1/(u + \sqrt{3})$. By the replacement rule, $\lim_{u\to\sqrt{3}} [(u - \sqrt{3})/(u^2 - 3)] = \lim_{u\to\sqrt{3}} [1/(u + \sqrt{3})]$. By the continuity of rational functions, the limit is $1/2\sqrt{3} = \sqrt{3}/6$.

17. We note that $f(x) = (x^2 - 4x + 3)/(x^2 - 2x - 3) = (x - 3)(x - 1)/(x - 3)(x + 1) = (x - 1)/(x + 1)$; therefore, by the replacement rule, $\lim_{x\to 3} f(x) = \lim_{x\to 3} [(x - 1)/(x + 1)]$. Then, by the continuity of rational functions, the limit becomes $2/4 = 1/2$.

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21. Note that $f(\Delta x) = [3(\Delta x)^2 + 2(\Delta x)]/\Delta x = 3(\Delta x) + 2$; therefore, by the replacement rule, $\lim_{\Delta x \to 0} f(\Delta x) = \lim_{\Delta x \to 0} (3(\Delta x) + 2) = 2$.

25. Since $\sqrt{x}$ is continuous, we have $\lim_{x \to 3} [(1 - \sqrt{x})(2 + \sqrt{x})] = \lim_{x \to 3} (2 - \sqrt{x} - x) = 2 - \sqrt{3} - 3 = -\sqrt{3} - 1$.

29. $|x - 1|$ is always positive. If $x > 1$, then $|x - 1|/(x - 1) = 1$. If $x < 1$, then $|x - 1|/(x - 1) = -1$. Since the function approaches -1 from the left and it approaches 1 from the right, no limit exists.

33. Divide by x/x , so $\lim_{x \to -\infty} [(2x - 1)/(3x + 1)] = \lim_{x \to -\infty} [(2 - 1/x)/(3 + 1/x)] = (2 - 0)/(3 + 0) = 2/3$. We used the fact $\lim_{x \to -\infty} (1/x) = 0$.

37. $\lim_{x \to 0} f(x) = 1 = f(0)$. $\lim_{x \to 1} f(x) = 2 \neq f(1)$, $f(1)$ is not defined. $\lim_{x \to 2} f(x)$ is not defined while $f(2) = 2$ is defined. $\lim_{x \to 3} f(x) = 1 \neq f(3)$, $f(3)$ is not defined. $\lim_{x \to 4} f(x)$ does not exist since no finite number is approached as x approaches 4 , while $f(4) = 2$.

41. By the replacement rule, $\lim_{x \to 0} (-x/x^3) = -\lim_{x \to 0} (1/x^2) = -\infty$. This is because $1/x^2$ is always positive.

45. By the replacement rule, $\lim_{u \to 0} [(u^3 + 2u^2 + u)/u] = \lim_{u \to 0} (u^2 + 2u + 1) = 1$.

49. Note that $f(x) = (x^2 - 5x + 6)/(x^2 - 6x + 8) = (x - 3)(x - 2)/(x - 4)(x - 2) = (x - 3)/(x - 4)$. By the replacement rule, $\lim_{x \to 2} f(x) = \lim_{x \to 2} [(x - 3)/(x - 4)] = (-1)/(-2) = 1/2$.

53. Expansion yields $\lim_{\Delta x \to 0} [(125 + 75\Delta x + 15(\Delta x)^2 + (\Delta x)^3 - 125)/\Delta x]$, and the replacement rule gives $\lim_{\Delta x \to 0} (75 + 15\Delta x + (\Delta x)^2) = 75$.

57. Factoring out $s - 3$ yields $\lim_{s \to 3} [(3s + 7)/(s - 3)]$. Since $s < 3$ causes the function to approach $-\infty$ and $s > 3$ causes the function to approach $+\infty$, no limit exists.

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61. (a) If the block of ice melts completely at time T, there is no ice, so neither the base nor the height exists, so $f(T)$ and $g(T)$ must both equal zero.

(b) The heat distribution need not be uniform throughout the room even though temperature is held constant. It could be the case that just before melting is completed, a very thin sheet of ice of area A is left. Then, $\lim_{t\to T} g(t) = 0 = g(T)$, but $\lim_{t\to T} f(t) = A \neq f(T)$. Another possibility is melting along the side faces only, with no melting in a vertical direction. The result is a tiny pillar of ice of height h just before melting is complete. Then, $\lim_{t\to T} f(t) = 0 = f(T)$, but $\lim_{t\to T} g(t) = h \neq g(T)$.

(c) Volume is given by the formula $f(t)\cdot g(t)$, so $\lim_{t\to T} [f(t)\cdot g(t)] = 0$.

(d) The product rule for limits states: $\lim_{t\to T} [f(t)\cdot g(t)] = \lim_{t\to T} f(t)\cdot \lim_{t\to T} g(t)$. The first example in part (b) gives the following result: $\lim_{t\to T} f(t)\cdot \lim_{t\to T} g(t) = A\cdot 0 = 0 = \lim_{t\to T} [f(t)\cdot g(t)]$. Similarly, the second example in part (b) gives: $\lim_{t\to T} f(t)\cdot \lim_{t\to T} g(t) = 0\cdot h = 0 = \lim_{t\to T} [f(t)\cdot g(t)]$

65. (a) Having shown that $\lim_{x\to x_0} [f_1(x) + f_2(x) + f_3(x)] = \lim_{x\to x_0} f_1(x) + \lim_{x\to x_0} f_2(x) + \lim_{x\to x_0} f_3(x)$, we let $g(x) = f_2(x) + f_3(x) + f_4(x)$. Now, $\lim_{x\to x_0} [f_1(x) + g(x)] = \lim_{x\to x_0} f_1(x) + \lim_{x\to x_0} g(x)$ by the basic sum rule. By using the extended sum rule for $n = 3$ on $\lim_{x\to x_0} g(x)$, we obtain our desired result.

(b) We let $g(x) = f_2(x) + f_3(x) + \ldots + f_{17}(x)$. Applying the extended sum rule for $n = 16$, we have $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} f_2(x) + \lim_{x\to x_0} f_3(x) + \ldots + \lim_{x\to x_0} f_{17}(x)$. By the basic sum rule, $\lim_{x\to x_0} [f_1(x) + f_2(x) + \ldots + f_{17}(x)] = \lim_{x\to x_0} [f_1(x) + g(x)] = \lim_{x\to x_0} f_1(x) + \lim_{x\to x_0} g(x)$, and substitution gives the desired result.

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65 (continued).

(c) Now, let $g(x) = f_2(x) + f_3(x) + \ldots + f_{n+1}(x)$. Apply the extended sum rule for $n \geqslant 2$, we have $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} f_2(x) = \lim_{x \to x_0} f_3(x) + \ldots + \lim_{x \to x_0} f_{n+1}(x)$. By the basic sum rule, $\lim_{x \to x_0} [f_1(x) + f_2(x) + \ldots + f_{n+1}(x)] = \lim_{x \to x_0} [f_1(x) + g(x)] = \lim_{x \to x_0} f_1(x) + \lim_{x \to x_0} g(x)$. Substitution shows that the extended sum rule holds for $n + 1$.

(d) In part (c), the statement holds for $n + 1$ assuming it holds for n . Earlier, we showed that the extended sum rule holds for $m = 3$ therefore, the extended sum rule is proven by induction.

SECTION QUIZ

1. Compute the limit:

   (a) $\lim_{x \to 3} |(x - 3)/(x - 3)|$

   (b) $\lim_{x \to 3} [(x - 3)/(x + 3)]$

   (c) $\lim_{x \to 3} ((x - 3)/|x - 3|)$

2. Find:

   (a) $\lim_{x \to 2} [(x - 2)/(x^3 - 8)]$

   (b) $\lim_{x \to \infty} [(x - 2)/(x^3 - 8)]$

   (c) $\lim_{x \to 1} [(x^2 - 1)/(2x^2 + x - 3)]$

   (d) $\lim_{x \to \infty} [(x^2 - 1)/(2x^2 + x - 3)]$

3. Let $f(x) = 2x^3 + 6x + 7$. Compute $f'(3)$.

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4. Sir Chuck, the clumsy knight, was fleeing from a fire breathing dragon when he suddenly noticed the castle wall before him. Fortunately for clumsy Chuck, Merlin whisked him into the castle as the dragon crashed snout-first into the wall. Chuck's position was $10t$ kilometers from the dragon's lair t hours after the dragon began chasing him. Earlier, clumsy Chuck had been quite still at the dragon's lair while trying to figure out how to get a dragon-skin coat. Let $y(t)$ be Chuck's position and $v(t)$ be his speed. Merlin saved Chuck at $t_0 = 1/2$.

   (a) Find $\lim_{t\to 0} y(t)$, $\lim_{t\to 0} v(t)$, $\lim_{t\to 1/4} y(t)$, and $\lim_{t\to 1/4} v(t)$.

   (b) If $v(t)$ suddenly became zero at $t_0 = 1/2$, does $\lim_{t\to 1/2} v(t)$ exist? Explain.

ANSWERS TO PREREQUISITE QUIZ

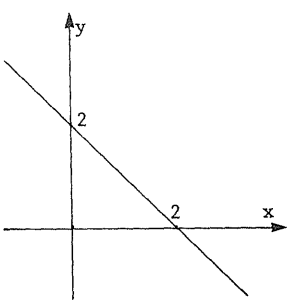
1. (a) $(x^2 + y^2)(x + y)(x - y)$

   (b) $(x - 2)(x^2 + 2x + 4)$

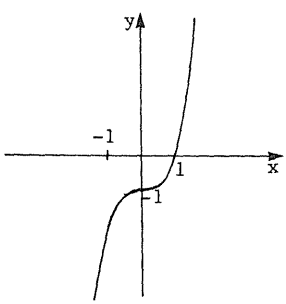
   (c) $4(x + 2)^2$

   (d) $(3x + 2)(x + 2)$

2. (a)



(b)



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ANSWERS TO SECTION QUIZ

1. (a) 1

   (b) 0

   (c) Does not exist

2. (a) 1/12

   (b) 0

   (c) 2/5

   (d) 1/2

3. 60

4. (a) $\lim_{t\to 0} y(t) = 0$ kilometers; $\lim_{t\to 0} v(t)$ does not exist because the speed "jumps" from 0 to 10 km/hr; $\lim_{t\to 1/4} y(t) = 5/2$ kilometers; $\lim_{t\to 1/4} v(t) = 10$ km/hr.

   (b) No, because the speed approaches 0 km/hr on one side of t_0 and it approaches 10 km/hr on the other side.

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1.3 The Derivative as a Limit and the Leibniz Notation

PREREQUISITES

1. Recall the $\Delta y/\Delta x$ method for computing derivatives (Section 1.1).
2. Recall formulas used to differentiate quadratic and linear functions (Section 1.1).
3. Recall the laws of limits (Section 1.2).

PREREQUISITE QUIZ

1. Use the $\Delta y/\Delta x$ method to differentiate $5x^2 + 3$.
2. (a) What is $\Delta y/\Delta x$ for $f(x) = 2x + 15$?

   (b) Use the result from part (a) to find $f'(3)$.
3. Differentiate the following functions:

   (a) $2 - 4x^2$

   (b) $5x$

   (c) $x^2 - 2x + 4$

   (d) $ax^2 + bx + c$, where $a,b,$ and c are constants.
4. Compute the following limits:

   (a) $\lim_{x\to 2}(x^2 - x)$

   (b) $\lim_{x\to 3}[(x^3 - 27)/(x - 3)]$

GOALS

1. Be able to state and to use the definition of the derivative.
2. Be able to interchange the Leibniz notation, dy/dx, with other notations for the derivative, such as $f'(x)$.

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STUDY HINTS

1. Derivative defined. It is defined as $\lim_{\Delta x \to 0} (\Delta y/\Delta x)$, where Δy is $f(x + \Delta x) - f(x)$. Previously, you saw the less precise form in Section 1.1 in which Δx was "replaced" by zero. Now, we take the limit of the difference quotient.

2. Continuity and differentiability. One implies the other; however, the reverse is not always true. What implication direction is true? (See Example 5.)

3. Nondifferentiability. A function is differentiable only if its graph is "smooth." A "pointed" function like the one in Fig. 1.3.2 is not differentiable at the "point."

4. Leibniz notation. dy/dx is just another notation for the derivative. It is not a fraction, so do not treat it as numerator and denominator; however, in many derivations, it acts like a fraction.

5. More notation. $(dy/dx)\big|_{x_0}$ means to evaluate the derivative at x_0.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The definition of the derivative is $f'(x) = \lim_{\Delta x \to 0} (\Delta y/\Delta x) = \lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/\Delta x\}$. Here, $\Delta y = [(x + \Delta x)^2 + (x + \Delta x)] - (x^2 + x) = [(x^2 + 2x\Delta x + (\Delta x)^2 + (x + \Delta x)] - (x^2 + x) = 2x\Delta x + (\Delta x)^2 + \Delta x$; $\Delta y/\Delta x = 2x + \Delta x + 1$. Thus, $\lim_{\Delta x \to 0} (\Delta y/\Delta x) = 2x + 1$.

5. The definition of the derivative is $f'(x) = \lim_{\Delta x \to 0} (\Delta y/\Delta x) = \lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/\Delta x\}$. Here, $\Delta y = 3/(x + \Delta x) - 3/x = [3x - 3(x + \Delta x)]/x(x + \Delta x) = -3\Delta x/x(x + \Delta x)$; $\Delta y/\Delta x = -3/x(x + \Delta x)$. Thus, $\lim_{\Delta x \to 0} (\Delta y/\Delta x) = -3/x^2$.

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9. The definition of the derivative is $f'(x) = \lim_{\Delta x \to 0} (\Delta y/\Delta x) = \lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/\Delta x\}$. Here, $\Delta y = 2\sqrt{x + \Delta x} - 2\sqrt{x} = 2(\sqrt{x + \Delta x} - \sqrt{x})$. Multiplying top and bottom by $\sqrt{x + \Delta x} + \sqrt{x}$ yields $\Delta y = 2\Delta x/(\sqrt{x + \Delta x} + \sqrt{x})$; $\Delta y/\Delta x = 2/(\sqrt{x + \Delta x} + \sqrt{x})$. Thus, $\lim_{\Delta x \to 0} (\Delta y/\Delta x) = 2/2\sqrt{x} = 1/\sqrt{x}$.

13. The difference quotient at $x_0 = 0$ is $[(1 + |0 + \Delta x|) - (1 + |0|)]/\Delta x = |\Delta x|/\Delta x$, which has no limit according to Example 5. Thus, the derivative at $x_0 = 0$ does not exist. On the other hand, $f(0) = 1$ and $\lim_{x \to 0} (1 + |x|) = \lim_{x \to 0} 1 + \lim_{x \to 0} |x| = 1$, as shown in Example 5. Since $\lim_{x \to 0} f(x) = f(0)$, $f(x)$ is continuous at $x_0 = 0$.

17. dy/dx is a new notation for the derivative. $\Delta y = [3(x + \Delta x)^3 + (x + \Delta x)] - (3x^3 + x) = [3x^3 + 9x^2\Delta x + 9x(\Delta x)^2 + 3(\Delta x)^3 + x + \Delta x] - (3x^3 + x) = 9x^2\Delta x + 9x(\Delta x)^2 + 3(\Delta x)^3 + \Delta x$, so $\Delta y/\Delta x = 9x^2 + 9x\Delta x + 3(\Delta x)^2 + 1$. Thus, $\lim_{\Delta x \to 0} (\Delta y/\Delta x) = dy/dx = 9x^2 + 1$.

21. The velocity is given by $f'(1)$, which is the limit of $[f(1 + \Delta t) - f(1)]/\Delta t$ as Δt approaches 0 . $[f(1 + \Delta t) - f(1)]/\Delta t = [5(1 + \Delta t)^3 - 5(1)^3]/\Delta t = [5(1 + 3\Delta t + 3(\Delta t)^2 + (\Delta t)^3) - 5]/\Delta t = 15 + 15\Delta t + 5(\Delta t)^2$. The limit is 15 , which is the velocity.

25. Using the result of Exercise 17, we have $(d/dx)(3x^3 + x)\big|_{x=1} = (9x^2 + 1)\big|_{x=1} = 10$.

29. The derivative is $\lim_{\Delta x \to 0} \{[1/(x + \Delta x)^2 - 1/x^2]/\Delta x\} = \lim_{\Delta x \to 0} \{[x^2 - (x^2 + 2x\Delta x + (\Delta x)^2)]/\Delta x(x + \Delta x)^2x^2\} = \lim_{\Delta x \to 0} [-(2x + \Delta x)/(x + \Delta x)^2x^2] = -2x/x^4 = -2/x^3$.

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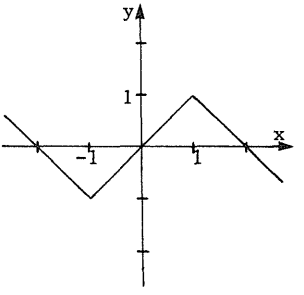
33. 

There is no unique answer. Following the result of Example 5, we wish to find a function with two "points" as shown in the figure. Such a function is:

$$f(x) = \begin{cases} -x - 2 & x \leq -1 \\ x & -1 \leq x \leq 1 \\ -x + 2 & x \geq 1 \end{cases} .$$

Two other possible answers are given in the back of the text.

SECTION QUIZ

1. State the definition of the derivative.
2. True or false: $|x|$ is not differentiable anywhere. Explain.
3. Evaluate $[(d/dx)(3x^2/4 - 5x)]\big|_3$.
4. Tough Tommy was enjoying his camping trip until the Thing woke him up. Upon seeing the Thing, tough Tommy almost jumped out of his pajamas. His position during the first hour is given by $x = 7t - t^2 - t^3$, where x is in miles and t is in hours.
   (a) Use the definition of the derivative to find his instantaneous speed, dx/dt .
   (b) Apply the definition of the derivative again to find his instantaneous acceleration, d^2x/dt^2 .

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ANSWERS TO PREREQUISITE QUIZ

1. $\Delta y/\Delta x = 10x + 5\Delta x$, so $f'(x) = 10x$.

2. (a) 2

   (b) 2

3. (a) $-8x$

   (b) 5

   (c) $2x - 2$

   (d) $2ax + b$

4. (a) 2

   (b) 27

ANSWERS TO SECTION QUIZ

1. The definition is $f'(x) = \lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/\Delta x\}$.

2. False; it is not differentiable only at $x = 0$. Elsewhere, $f'(x) = -1$ if $x < 0$ and $f'(x) = 1$ if $x > 0$.

3. $-1/2$

4. (a) $7 - 2t - 3t^2$ miles/hour

   (b) $-2 - 6t$ miles/hour^2

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1.4 Differentiating Polynomials

PREREQUISITES

1. Recall how to compute derivatives using the limit method (Section 1.3).

PREREQUISITE QUIZ

1. Describe the limit method for computing derivatives.

GOALS

1. Be able to combine the power rule, the sum rule, and the constant multiple rule to differentiate polynomials.

STUDY HINTS

1. Basic differentiation rules. Memorize $(d/dx)x^n = nx^{n-1}$, $(kf)'(x) = kf'(x)$, and $(f + g)'(x) = f'(x) + g'(x)$.
2. Polynomial rule. This rule incorporates the three basic rules introduced in this section. Don't memorize it, but you should be able to use the three basic rules to differentiate any polynomial.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the power rule, $(d/dx)x^{10} = 10x^9$.
5. By the power and constant mulitple rules, $(d/dx)(-5x^4) = 5(4x^3) = -20x^3$.
9. By the power and constant mulitple rules, $(d/dx)(3\sqrt{x}) = (3(1/2)(x^{-1/2}) = 3/2\sqrt{x}$.
13. $f(x) + g(x) = 3x^2 + x + 13$, so $(f + g)'(x) = 6x + 1$. On the other hand, $f'(x) = g'(x) = (6x) + (1)$.

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17. By the sum rule, the derivative is $f'(x) - [2g(x)]'$. By the constant multiple rule, this is $f'(x) - 2g'(x)$.

21. By the polynomial rule, the derivative is $5t^4 + 12t + 8$.

25. By the polynomial rule, $g'(s) = 13s^{12} + 96s^7 - 21s^6/8 + 4s^3 + 4s^2$.

29. By the polynomial rule, $g'(h) = 80h^9 + 9h^8 - 113h$.

33. Multiplying out, $(t^3 - 17t + 9)(3t^5 - t^2 - 1) = 3t^8 - 51t^6 + 26t^5 + 16t^3 - 9t^2 + 17t - 9)$. Thus, the polynomial rule yields $f'(t) = 24t^7 - 306t^5 + 130t^4 + 48t^2 - 18t + 17$.

37. Multiplication gives $h(t) = t^7 - t^5 + 9t^3 - 9t$, so the polynomial rule gives $h'(t) = 7t^6 - 5t^4 + 27t^2 - 9$.

41. By the sum rule and Example 4, Section 1.3, $f'(x) = 2x - 1/2\sqrt{x}$.

45. Multiplication gives $f(x) = 1 - x$, so $f'(x) = -1$.

49. The slope of the tangent line is $(d/dx)(x^4 - x^2 + 3x)\big|_1 = (4x^3 - 2x + 3)\big|_1 = 5$.

53. Starting with $(d/dx)x^n = nx^{n-1}$, we guess that the desired function is x^{n+1} . Differentiating this gives $(n + 1)x^n$. So we now guess $x^{n+1}/(n+1)$, whose derivative is x^n . Noting that the derivative of a constant is zero, our desired function is $x^{n+1}/(n+1) + C$, where C is any constant.

57. Recall from basic geometry that $V(r) = 4\pi r^3/3$ and that the surface area is $A(r) = 4\pi r^2$. (These formulas are on the inside front cover of the text.) By the constant multiple and power rules, $V'(r) = (4\pi/3)(3r^2) = 4\pi r^2 = A(r)$.

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SECTION QUIZ

1. In each case, compute dy/dx.

   (a) $y = x^{91} - x^{53} + x$

   (b) $y = -4x^6 + 8x^3$

   (c) $y = 2x^5(x^2 + 1)$

   (d) $y = a^2 + a$, a is constant

2. Find the equation of the line perpendicular to the curve $y = x^4/4 - 2x^3 + 3x - 1$ at $x_0 = 2$.

3. (a) Compute du/dy for $u = y^5 - y^3/2 + y$.

   (b) Compute $g'(1)$ for $g(t) = 5t^5 + 4t^4 + 3t^3 + 2t^2 + t + 1$.

4. Your hillbilly relatives are visiting the city for the first time. At the local department store, your young cousin tries to run up the down escalator. Horizontally, the escalator moves 30 feet in 10 seconds. Since your cousin tires as he runs up, his position on the nonmoving escalator with respect to the ground is given by $x = t^2 - t^3/6 - t^4/40$ for $0 \leqslant t \leqslant 3$. Answer the following questions with respect to the ground.

   (a) Compute the speed of the escalator.

   (b) Compute the speed of your hillbilly cousin on the stationary escalator.

   (c) Use your answers in parts (a) and (b) to compute his speed on the moving escalator

ANSWERS TO PREREQUISITE QUIZ

1. Compute the difference quotient $\Delta y/\Delta x = [f(x + \Delta x) - f(x)]/\Delta x$.
   Simplify as much as possible and then take the limit as Δx approaches 0 .

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ANSWERS TO SECTION QUIZ

1. (a) $91x^{90} - 53x^{52} + 1$

   (b) $-24x^5 + 24x^2$

   (c) $14x^6 + 10x^4$

   (d) 0

2. $13y - x + 93 = 0$

3. (a) $5y^4 - 3y^2/2 + 1$

   (b) 55

4. (a) 3 feet/second .

   (b) $(2t - t^2/2 - t^3/10)$ feet/second .

   (c) $(2t - t^2/2 - t^3/10 - 3)$ feet/second .

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1.5 Products and Quotients

PREREQUISITES

1. Recall how to compute limits (Section 1.3).
2. Recall how to differentiate polynomials (Section 1.4).

PREREQUISITE QUIZ

1. What is $\lim_{x\to 0} [(x + 5)(x^2 + 3x)/x]$?
2. Differentiate the following functions:
   (a) $f(x) = x^{50} + 30x^{25} + x^4 + 50$.
   (b) $f(x) = 3x^{70} - 2x^{48} + 5(10)^6$.

GOALS

1. Be able to state and apply the product rule.
2. Be able to state and apply the quotient rule.

STUDY HINTS

1. Product rule. Memorize the formula $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$. You will probably not be required to reproduce its derivation. You're wasting time if you're using this rule when one of f or g is constant; use the constant multiple rule.
2. Quotient rule. Memorize the formula $(f/g)'(x) = [f'(x)g(x) - f(x)g'(x)]/[g(x)]^2$. The memory aid preceding Example 5 may be useful to you. As with the product rule, don't be too concerned with the derivation of the quotient rule.
3. Reciprocal rule. An alternative to learning this rule is to apply the quitient rule with $f(x) = 1$; however, you will find that the reciprocal rule is faster to use.

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4. Integer power rule. This is just an extension of the rule presented in Section 1.4. The rule you learned earlier now holds for any integer. CAUTION: A very common mistake is to add 1 to the exponent rather than subtracting 1 . Thus, many students equate $d(x^{-2})/dx$ with $-2x^{-1}$ rather than the correct answer, $-2x^{-3}$. Be careful!

5. Differentiation rules. At this point in your studies, you should know the rules for differentiating a power, a sum, a constant multiple, a product, and a quotient. All of the others are special cases of these rules and need not be memorized.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the product rule, $(d/dx)[(x^2 + 2)(x + 8)] = (x^2 + 2)(1) + (2x)(x + 8) = 3x^2 + 16x + 2$. On the other hand, $(x^2 + 2)(x + 8) = x^3 + 8x^2 + 2x + 16$, so the derivative is $3x^2 + 16x + 2$.

5. By the product rule, $(d/dx)[(x^2 + 2x + 1)(x - 1)] = (x^2 + 2x + 1)(1) + (2x + 2)(x - 1) = 3x^2 + 2x - 1$. On the other hand, $(x^2 + 2x + 1)(x - 1) = x^3 + x^2 - x - 1$, so the derivative is $3x^2 + 2x - 1$.

9. By the product rule, $(d/dx)[(x - 1)(x^2 + x + 1)] = (x - 1)(2x + 1) + (1)(x^2 + x + 1) = 3x^2$. On the other hand, $(x - 1)(x^2 + x + 1) = x^3 - 1$, so the derivative is $3x^2$.

13. $d(x^{5/2})/dx = (d/dx)(x^2 \cdot \sqrt{x}) = 2x\sqrt{x} + x^2/2\sqrt{x} = 5x^{3/2}/2$.

17. By the quotient rule, $(d/dx)[(x - 2)/(x^2 + 3)] = [(1)(x^2 + 3) - (x - 2)(2x)]/(x^2 + 3)^2 = (-x^2 + 4x + 3)/(x^2 + 3)^2$.

21. By the quotient rule, $(d/dx)[(x^2 + 2)/(x^2 - 2)] = [(2x)(x^2 - 2) - (x^2 + 2)(2x)]/(x^2 - 2)^2 = -8x/(x^2 - 2)^2$.

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25. By the quotient rule, $(d/dr)[(r^2 + 2)/r^8] = [(2r)(r^8) - (r^2 + 2)(8r^7)]/r^{16} = (-6r^9 - 16r^7)/r^{16} = (-6r^2 - 16)/r^9$.

29. We expand, first, to get $((x^2 + 1)^2 + 1)/((x^2 + 1)^2 - 1) = (x^4 + 2x^2 + 2)/(x^4 + 2x^2)$. By the quotient rule, the derivative is $[(4x^3 + 4x)(x^4 + 2x^2) - (x^4 + 2x^2 + 2)(4x^3 + 4x)]/(x^4 + 2x^2)^2 = (-8x^3 - 8x)/(x^4 + 2x^2)^2$.

33. By the reciprocal rule, $(d/dx)[1/(x + 1)^2] = (d/dx)[1/(x^2 + 2x + 1)] = -(2x + 2)/(x^2 + 2x + 1)^2 = -2(x + 1)/(x + 1)^4 = -2/(x + 1)^3$.

37. Combining the reciprocal rule with other derivative rules, we get $(d/dy)[4(y + 1)^2 - 2(y + 1) - 1/(y + 1)] = (d/dy)[4(y^2 + 2y + 1)] - 2 + 1/(y + 1)^2 = 8y + 6 + (y + 1)^{-2}$.

41. It is easiest to mulitply first to get $(3\sqrt{x} + 1)x^2 = 3x^{5/2} + x^2$. Then, the derivative is $15x^{3/2}/2 + 2x$. On the other hand, the product rule may be used to obtain $(3/2\sqrt{x})x^2 + (3\sqrt{x} + 1)(2x) = 15x^{3/2}/2 + 2x$. Both methods give the same result.

45. By the reciprocal rule, $(d/dx)[\sqrt{2}/(1 + 3\sqrt{x})] = -\sqrt{2}(3/2\sqrt{x})/(1 + 3\sqrt{x})^2 = -3/\sqrt{2x}(1 + 3\sqrt{x})^2$.

49. By the reciprocal rule, $f'(x) = -(1/2\sqrt{x})/(\sqrt{x})^2 = -1/2x^{3/2}$. Thus, the slope of the tangent line at $x = 2$ is $(-1/2x^{3/2})|_2 = -1/4\sqrt{2} = -\sqrt{2}/8$.

53. By the product and sum rules, the derivative is $[f(x) + xf'(x)] + g'(x) = (4x^5 - 13x) + x(20x^4 - 13) + (3x^2 + 2) = 24x^5 + 3x^2 - 26x + 2$.

57. We apply the reciprocal rule to get $(d/dx)(1/P(x)) = -P'(x)/[P(x)]^2$. Let $P(x) = ax^2 + bx + c$, so $-P'(x) = -2ax - b = 0$. The only solution is $x = -b/2a$ if $P(-b/2a) \neq 0$. $(d/dx)(1/P(x))$ is never zero when $P(x)$ is a perfect square such as x^2 or $x^2 + 2x + 1$.

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SECTION QUIZ

1. By the quotient rule, $(d/dx)(2/x^3) = [(0)(x^3) - (2)(3x^2)]/(x^3)^2 = -6x^2/x^6 = -6x^{-4}$. On the other hand, $2/x^3 = 2x^{-3}$, so by the integer power rule, $(d/dx)(2x^{-3}) = (2)(-3x^{-2}) = -6x^{-2}$. Why aren't the two derivatives equal?

2. Suppose $(d/dy)\omega = \Omega$, $(d/dy)\gamma = \Gamma$, and $(d/dy)\delta = \Delta$. What is $(d/dy)[(\omega\gamma - \gamma\delta)/\omega\delta]$?

3. Differentiate $(x^2 + 2x + \sqrt{x})(x^2 + 3)/(x^2 - 2)(x + \sqrt{x})$.

4. The mad scientist, Liver Louie, enjoys eating chicken livers. After performing numerous genetic experiments, he can produce chickens with $2x - 1$ livers in the xth generation. Also, he has increased the size of each liver so that each weighs $x^2 + x + 1$ ounces.

   (a) Use the product rule to determine how fast liver weight increases in each chicken during the xth generation.

   (b) If total revenue, R, from one chicken's liver is $x^2 + 3x + 1$ dollars, what is the inflation rate for chicken livers in the xth generation, i.e., what is $(d/dx)(R/Q)$ where Q is one chicken's liver weight?

ANSWERS TO PREREQUISITE QUIZ

1. 15

2. (a) $50x^{49} + 750x^{24} + 4x^3$

   (b) $210x^{69} + 96x^{47}$

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ANSWERS TO SECTION QUIZ

1. By the integer power rule, the derivative of x^{-3} is $-3x^{-4}$, not $-3x^{-2}$.

2. $[(\Omega\gamma + \omega\Delta - \Gamma\delta - \gamma\Delta)\omega\delta - (\omega\gamma - \gamma\delta)(\Omega\delta + \omega\Delta)]/\omega^2\delta^2$

3. $\{[(2x + 2 + 1/2\sqrt{x})(x^2 + 3) + (x^2 + 2x + \sqrt{x})(2x)](x^2 - 2)(x + \sqrt{x}) - (x^2 + 2x + \sqrt{x})(x^2 + 3)[(2x)(x + \sqrt{x}) + (x^2 - 2)(1 + 1/2\sqrt{x})]\}/(x^2 - 2)^2(x + \sqrt{x})^2$

4. (a) $6x^2 + 2x + 1$

   (b) $[(2x + 3)(2x - 1)(x^2 + x + 1) - (6x^2 + 2x + 1)(x^2 + 3x + 1)]/(2x - 1)^2(x^2 + x + 1)^2$

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1.6 The Linear Approximation and Tangent Lines.

PREREQUISITES

1. Recall the point-slope form of the line (Section R.4).
2. Recall the relationship between $f'(x)$ and $\Delta y/\Delta x$, i.e., recall the definition of the derivative (Section 1.3).

PREREQUISITE QUIZ

1. What is the equation of a line with slope m and passing through the point (x_0, y_0) ?
2. How is $f'(x)$ related to $\Delta y/\Delta x$?

GOALS

1. Be able to write the equation of the tangent line to a curve at a given point.
2. Be able to find an approximation by using derivatives.

STUDY HINTS

1. Tangent lines. The equation doesn't need to be memorized: if you know $y = y_0 + m(x - x_0)$, just substitute $y_0 = f(x_0)$ and $m = f'(x_0)$.
2. Linear approximation. You should either memorize or learn to derive the formula. If you remember that $[f(x_0 + \Delta x) - f(x_0)]/\Delta x \approx f'(x_0)$ from Section 1.1, you can easily derive the formula by multiplying through by Δx and then adding $f(x_0)$ to both sides. Note that Δx may be negative. Using a calculator defeats the purpose of learning what the linear approximation is all about.

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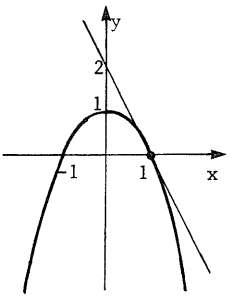
3. Relationship between tangency and the approximation. Notice that the linear approximation was derived from $\Delta y/\Delta x \approx f'(x_0)$. Also notice that the tangent line may be rewritten as $m = (y - y_0)/(x - x_0)$ or $f'(x_0) = \Delta y/\Delta x$. Thus, one basic formula gives you the concepts of this section.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

The tangent line to $y = f(x)$ is $y = f(x_0) + f'(x_0)(x - x_0)$. For $y = 1 - x^2$, $f'(x) = -2x$, so $y = 0 + (-2)(x - 1) = -2x + 2$.

5. The tangent line to $y = f(x)$ is $y = f(x_0) + f'(x_0)(x - x_0)$. Combining the product and quotient rules, we have $f'(x) = (2x)[3x/(x + 2)] + (x^2 - 7)[3(x + 2) - 3x(1)]/(x + 2)^2$, so $f'(x_0) = 0[0/2] + (-7)[(3)(2) - (0)(1)]/(2)^2 = -21/2$. Also, $f(x_0) = -7(0/2) = 0$; therefore, the tangent line is $y = (-21/2)x$.

9. By the quotient rule, $f'(x) = [(1)(x + 1) - (x)(1)]/(x + 1)^2 = 1/(x + 1)^2$ and $f'(x_0) = 1/4$. Since $f(x_0) = 1/2$, the tangent line is $y = 1/2 + (1/4)(x - 1)$. To find the x-intercept, set $y = 0$. This gives $x = -1$ as the crossing point.

13. Use the approximation $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$. Let $f(x) = x^2$, so $f'(x) = 2x$. Thus, $f(2.02) = f(2 + 0.02) \approx f(2) + f'(2)(0.02) = 4 + 4(0.02) = 4.08$. The exact value is 4.0804 .

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17. Use the approximation $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$. Let $f(x) = \sqrt{x}$, so $f'(x) = 1/2\sqrt{x}$. Thus, $f(16.016) = f(16 + 0.016) \approx f(16) + f'(16)(0.016) = 4 + (1/8)(0.016) = 4.002$. The exact value is approximately 4.001999 .

21. Using the approximation $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$, we let $f(x) = x^4$, so $f'(x) = 4x^3$. Thus, $f(2.94) = f(3 - 0.06) \approx f(3) + f'(3)(-0.06) = 81 + 108(-0.06) = 74.52$.

25. Let $A(r) = \pi r^2$, so $A'(r) = 2\pi r$. $A(3.04) = A(3 + 0.04) \approx A(3) + A'(3)(0.04)$. Therefore, the approximate increase in area is $A'(3)(0.04) = 6\pi(0.04) = 0.24\pi$.

29. We use $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$ with $x_0 = 3$ and $\Delta x = 0.023$. Thus, $f(x_0) = (12)(5) = 60$ and $f'(x) = (2x)(x + 2) + (x^2 + 3)(1)$, and so $f'(x_0) = (6)(5) + (12)(1) = 42$. Therefore, $f(3.023) \approx 60 + 42(0.023) = 60.966$.

33. The tangent line has the equation $y = f(1) + f'(1)(x - 1)$. Here, $f'(x) = 8x^7 + 4x$, so $f'(1) = 12$. Since $f(1) = 4$, the equation is $y = 4 + 12(x - 1)$ or $y = 12x - 8$.

37. We use $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$ with $x_0 = 1$ and $\Delta x = -0.0003$. Here, $f'(s) = 4s^3 - 15s^2 + 3$, so $f'(1) = 4 - 15 + 3 = -8$. Also, $f(1) = 1 - 5 + 3 - 4 = -5$, so $f(0.9997) \approx -5 + (-8)(-0.0003) = -4.9976$.

41. For each case, we use $h(t_0 + \Delta t) \approx h(t_0) + f'(t_0)\Delta t$, where $h'(t) = -8t + 7$.

$h(3.001) = h(3 + 0.001) \approx h(3) + h'(3)(0.001) = -14.25 + (-17)(0.001)$
$= -14.267$.

$h(1.97) = h(2 - 0.03) \approx h(2) + h'(2)(-0.03) = -1.25 + (-9)(-0.03) = -0.98$.

$h(4.03) = h(4 + 0.03) \approx h(4) + h'(4)(0.03) = -35.25 + (-25)(0.03) = -36.00$.

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45. (a) $f'(x) = 4x^3$ and the linear approximation to $f(x)$ near $x = 2$ is $f(2) + f'(2)\Delta x = (2)^4 + 4(2)^3\Delta x = 16 + 32\Delta x$.

(b) In general, the linear approximation at $x_0 + \Delta x$ is $x_0^4 + 4x_0^3\Delta x$. The actual value is $(x_0 + \Delta x)^4 = x_0^4 + 4x_0^3\Delta x + 6x_0^2(\Delta x)^2 + 4x_0(\Delta x)^3 + (\Delta x)^4$. Thus, the difference between the linear approximation and the actual value is $-[6x_0^2(\Delta x)^2 + 4x_0(\Delta x)^3 + (\Delta x)^4] = -(\Delta x)^2[6x_0^2 + 4x_0\Delta x + (\Delta x)^2] = -(\Delta x)^2[(2x_0 + \Delta x)^2 + 2x_0^2]$, which is ≤ 0 . Thus, the linear approximation is smaller than the actual value.

(c) We want $6x_0^2(\Delta x)^2 + 4x_0(\Delta x)^3 + (\Delta x)^4 \leq 1.6$, where $x_0 = 2$, i.e., $24(\Delta x)^2 + 8(\Delta x)^3 + (\Delta x)^4 \leq 1.6$. Therefore, the interval is $[1.730, 2.247]$.

49. The following chart includes a few examples to verify that $1 - x + x^2$ is indeed a better approximation of $1/(1 + x)$ than $1 - x$.

x	$1/(1 + x)$	$1 - x$	$1 - x + x^2$
0.1	0.9090909091	0.9	0.91
0.01	0.9900990099	0.99	0.9901
0.001	0.999000999	0.999	0.999001
0.0001	0.99990001...	0.9999	0.99990001

The above calculations demonstrate that an extra term will provide a better approximation. By long division, the first three terms of $3600/(60 + x)$ are $(60 - x + x^2/60)$ miles/hour .

SECTION QUIZ

1. Find an approximate value for $3(2.008)^2 - 5(2.008) + 3$.

2. A cube has edges 4.989 cm. long. Estimate its surface area.

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3. Consider the graph of the equation $y = x^3 + 2x - 1$

   (a) What is the tangent line at $x = 2$.

   (b) Where does the tangent line at $x = -1$ intersect the tangent line in part (a)?

4. Cathy Cuisine enjoys dining at the city's finest gourmet restaurants. However, she never carries cash and somebody had just stolen all of her National Rapids money orders. The furious restaurateur questions her, "What will you do? What will you do? Do you want to be dumped in the slammer or wash $x^3/100 + x + 65$ dishes for each x dollars you owe us?"

   (a) Use the $\Delta y/\Delta x$ method to compute dy/dx for $y = x^3/100 + x + 65$.

   (b) Suppose Cathy's dinner cost \$35 . Use the linear approximation with $x_0 = 30$ to estimate how many dishes she must wash.

   (c) Use $x_0 = 40$ to find another approximation.

   (d) If she washes 100 dishes/hour , about how many hours must she wash dishes?

ANSWERS TO PREREQUISITE QUIZ

1. $y = y_0 + m(x - x_0)$
2. $f'(x) = \lim_{\Delta x \to 0} (\Delta y/\Delta x)$

ANSWERS TO SECTION QUIZ

1. 5.056
2. The surface area is $6x^2$, so $6(4.989)^2 \approx 149.34$ cm^2 .
3. (a) $y = 14x - 17$

   (b) (2 , 11)

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4. (a) $3x^2/100 + 1$

(b) 505

(c) 500

(d) About 5 to 5.05 hours

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1.R Review Exercises for Chapter 1

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the quadratic function rule, $f'(x) = 2x$.

5. By the quadratic function rule with $a = 0$, $f'(x) = 2$.

9. By the polynomial rule, $f'(x) = -50x^4 + 24x^2$.

13. Combining the sum rule and the rule for square roots, $f'(x) = 9x^2 - 1/\sqrt{x}$.

17. By the quotient rule, $f'(x) = [2x(x^2 - 1) - (x^2 + 1)(2x)]/(x^2 - 1)^2 = -4x/(x^2 - 1)^2$.

21. First, expand $(s + 1)^2$ and then, by the product rule, $(d/ds)[(s + 1)^2 \cdot (\sqrt{s} + 2)] = (d/ds)[(s^2 + 2s + 1)(\sqrt{s} + 2) = (2s + 2)(\sqrt{s} + 2) + (s^2 + 2s + 1)(1/2\sqrt{s}) = (5s^2 + 8s^{3/2} + 6s + 8\sqrt{s} + 1)/2\sqrt{s}$.

25. By the reciprocal rule, $(d/dt)(3t^2 + 2)^{-1} = -6t/(3t^2 + 2)^2$.

29. Multiplying and using the reciprocal rule, $(d/dx)[1/\sqrt{x}(\sqrt{x} - 1)] = (d/dx)[1/(x - \sqrt{x})] = (1 - 1/2\sqrt{x})/(x - \sqrt{x})^2 = -(2\sqrt{x} - 1)/2\sqrt{x}(x - \sqrt{x})^2$.

33. Factor out $(x - 1)$ to get $\lim_{x\to 1}[(x^3 - 1)/(x - 1)] = \lim_{x\to 1}[(x - 1)(x^2 + x + 1)/(x - 1)] = \lim_{x\to 1}(x^2 + x + 1) = 3$.

37. By the continuity of rational functions, $\lim_{x\to 3}[(3x^2 + 2x)/x] = (27 + 6)/3 = 11$.

41. We recognize the limit as the definition of the derivative, so the limit is $f'(x) = 4x^3 + 6x$.

45. Divide by x^3/x^3 to get $\lim_{x\to\infty}[(5/x + 4/x^3)/(5 + 9/x^2)]$. Now, using the fact that $\lim_{x\to\infty}(1/x) = 0$, the limit becomes $0/5 = 0$.

49. The definition of the derivative is $\lim_{\Delta x\to 0}\{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\}$, so $f'(1) = \lim_{\Delta x\to 0}\{[f(1 + \Delta x) - f(1)]/\Delta x\} = \lim_{\Delta x\to 0}\{[3(1 + \Delta x)^3 + 8(1 + \Delta x) - 3(1)^3 - 8(1)]/\Delta x\} = \lim_{\Delta x\to 0}\{[9\Delta x + 9(\Delta x)^2 + 3(\Delta x)^3 + 8\Delta x]/\Delta x\} = \lim_{\Delta x\to 0}[17 + 9\Delta x + 3(\Delta x)^2] = 17$.

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53. The slope of the tangent line is $f'(x_0)$. For $y = x^3 - 8x^2$, $f'(x) = 3x^2 - 16x$, and so the slope is $f'(1) = -13$.

57. The slope of the tangent line is $f'(x_0)$. For $y = 3x^4 - 10x^9$, $f'(x) = 12x^3 - 90x^8$, and so the slope is $f'(0) = 0$.

61. The velocity is given by $f'(t_0)$. By the quotient rule, $f'(t) = [(2t + 1/2\sqrt{t})(1 + \sqrt{t}) - (t^2 + \sqrt{t})(1/2\sqrt{t})]/(1 + \sqrt{t})^2$. At $t_0 = 1$, the velocity is $f'(1) = [(5/2)(2) - 2(1/2)]/(2)^2 = 1$ meter per second.

65. The linear approximation is $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$. Let $f(x) = \sqrt{x}$, so $f'(x) = 1/2\sqrt{x}$. Thus, $f(4.0001) = f(4 + 0.0001) \approx f(4) + f'(4)(0.0001) = 2 + (1/4)(0.0001) = 2.000025$.

69. The linear approximation is $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$. $h'(s) = 12s^2 - 4s^3$, so $h(2.95) = h(3 - 0.05) \approx h(3) + h'(3)(-0.05) = 27 + (0)(-0.05) = 27.00$.

73. The tangent line is given by $y = y_0 + f'(x_0)(x - x_0)$. By the quotient rule, $f'(x) = [(3x^2)(x^3 + 11) - (x^3 - 7)(3x^2)]/(x^3 + 11)^2$ and $f'(2) = [(12)(19) - (1)(12)]/(19)^2 = 216/361$, so the tangent line is $y = (1/19) + (216/361)(x - 2) = (216x - 431)/361$.

77. By the sum rule, the derivative is $f'(x) + g'(x) = (4x - 5) + (3x/2 + 2) = 11x/2 - 3$, so at $x = 1$, the derivative is $5/2$.

81. The volume V is $(r)(r)(3r) = 3r^3$. The surface area is $A = 2(r)(r) + 4(r)(3r) = 14r^2$. Now, $dV/dr = 9r^2$, which is $9/14$ of A .

85. The slope of the tangent is $y' = 2x - 2$. The tangency point is $(x , x^2 - 2x + 2)$, so the slope of the line passing through the origin is $(x^2 - 2x + 2)/x = 2x - 2$. The equation reduces to $x^2 - 2 = 0$, so x must be $+\sqrt{2}$. $x = -\sqrt{2}$ is not acceptable because we want a positive slope. Therefore, the tangent line is $y = 2 - 2\sqrt{2} + 2 + (2\sqrt{2} - 2)(x - \sqrt{2}) = (2\sqrt{2} - 2)x$.

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89. (a) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$, and $g(x)\ b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0$, $b_m \neq 0$. Then $f(x)g(x) = a_n b_m x^{n+m}$ + lower degree terms. Since $a_n \neq 0$ and $b_m \neq 0$, $a_n b_m \neq 0$ and so, $\deg[f(x)g(x)] = n + m$. Also, $\deg[f(x)] + \deg[g(x)] = n + m$.

(b) Let $f(x) = (a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0)/(b_k x^k + b_{k-1}x^{k-1} + \ldots + b_0)$, $a_n \neq 0$ and $b_k \neq 0$, and $g(x) = (c_m x^m + c_{m-1}x^{m-1} + \ldots + c_0)/d_\ell x^\ell + d_{\ell-1}x^{\ell-1} + \ldots + d_0)$, $c_m \neq 0$ and $d_\ell \neq 0$. Then $f(x)g(x) = (a_n c_m x^{n+m}$ + lower degree terms$)/(b_k d_\ell x^{k+\ell}$ + lower degree terms). By definition, $\deg[f(x)g(x)] = n + m - k - \ell$. Also, $\deg[f(x)] = n - k$ and $\deg[g(x)] = m - \ell$, so $\deg[f(x)] + \deg[g(x)] = n + m - k - \ell$.

(c) Define $f(x)$ as in part (b). Then $f'(x) = [(na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1)\ .\ (b_k x^k + b_{k-1}x^{k-1} + \ldots + b_0) - (a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0)\ \cdot\ (kb_k x^{k-1} + (k-1)b_{k-1}x^{k-2} + \ldots + b_1)]/(b_k x^k + b_{k-1}x^{k-1} + \ldots + b_0)^2 = [(na_n b_k x^{n+k-1}$ + lower degree terms$) - (ka_n b_k x^{n+k-1}$ + lower degree terms$)]/(b_k^2 x^{2k}$ + lower degree terms$) = ((n-k)a_n b_k x^{n+k-1}$ + lower degree terms$)/(b_k^2 x^{2k}$ + lower degree terms). Therefore, $\deg[f'(x)] = (n + k - 1) - (2k) = n - k - 1$. Also, $\deg[f(x)] - 1 = n - k - 1$. This assumes $n - k \neq 0$.

If $\deg[f(x)] = 0$, i.e., $n - k = 0$, then the numerator in $f'(x)$ has degree at most $n + k - 2$. The denominator has degree $2k$, so $\deg[f'(x)]$ is at most $(n + k - 2) - 2k = n - k - 2 = -2$. This is all that can be said, as the example $f(x) = (x^k + 1)/(x^k - 1)$ shows.

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TEST FOR CHAPTER 1

1. True or false:

   (a) If $f'(x) = g'(x)$, then $f(x) = g(x)$.

   (b) $\lim_{x \to 1} [(x^2 - 1)/(x - 1)] = 0$.

   (c) If $r = s^5 + 5s^4$, then $ds/dr = 5s^4 + 20s^3$.

   (d) For any constant b , the derivative, with respect to x , of $bf(x)$ is $bf'(x)$.

   (e) If $y = (x^3 + x^2)x$, then $(dy/dx)|_2$ can only be 44 .

2. (a) State the product rule.

   (b) State the quotient rule.

3. Differentiate $(x^3 + 3x + 2)(5x^4 - 3x^3 + x)/(x^2 + 5)$. Do not simplify.

4. (a) State the definition of the derivative.

   (b) Use the limit method to find the derivative of $x/(x + 3)$.

5. Find $(dy/dx)|_1$ if $y = (3x^2 + x)/(x + 5)$.

6. Let $h(z) = 5z^7 + 2z^5 - z^4 + 3z^3 - 11z + 6$.

   (a) Find the tangent line at $z_0 = 1$.

   (b) Find the line perpendicular to $h(z)$ at $z_0 = 1$.

   (c) If h is in miles and z is in hours, what is the physical interpretation of dh/dz ?

   (d) Find an approximation for $h(0.97)$.

7. Find $(d/dt)[t^3/(t^2 + 1)]$.

8. Differentiate the following functions in x :

   (a) $f(x) = (x^6 - x^4 + 2)(3x^3 + x^2 - 3x)$

   (b) $f(x) = (x^4 + x)/(2x^4 - x^3 + x - 1)$

   (c) $f(x) = (x^5 + x^4 + x^3)^{-1}$

9. Let $f(x) = \begin{cases} x^3 & \text{if } x \leq 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$. Use limits to determine if $f'(0)$ exists.

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10. One day in prehistory, a caveman, whose cave entrance was located at (1,3) of a parabolic hill described by $y = 4 - x^2$, teased a pterydactyl flying overhead. The angry reptile flew down at a tangent to the hill and barely skimmed the cave entrance.
    (a) What equation describes the flight path?
    (b) Where should a fire be built at $y = 0$ if the caveman wants roast pterydactyl for dinner?

ANSWERS TO CHAPTER TEST

1. (a) False, $f(x)$ and $g(x)$ may differ by a constant.
   (b) False, the limit is 2.
   (c) False, $dr/ds = 5s^4 + 20s^3$, not ds/dr.
   (d) True
   (e) True
2. (a) The derivative of $f(x)g(x)$ is $f'(x)g(x) + f(x)g'(x)$.
   (b) The derivative of $f(x)/g(x)$ is $[f'(x)g(x) - f(x)g'(x)]/[g(x)]^2$, provided $g(x) \neq 0$.
3. $[((3x^2 + 3)(5x^4 - 3x^3 + x) + (x^3 + 3x + 2)(20x^3 - 9x^2 + 1))(x^2 + 5) - (x^3 + 3x + 2)(5x^4 - 3x^3 + x)(2x)]/(x^2 + 5)^2$.
4. (a) $f'(x) = \lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/\Delta x\}$.
   (b) $3/(x + 3)^2$
5. 19/18
6. (a) $h(z) = 39z - 35$
   (b) $h(z) = 4 - (z - 1)/39$
   (c) It is the velocity.
   (d) 2.83

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7. $(t^4 + 3t^2)/(t^2 + 1)^2$

8. (a) $(6x^5 - 4x^3)(3x^3 + x^2 - 3x) + (x^6 - x^4 + 2)(9x^2 + 2x - 3)$

   (b) $[(4x^3 + 1)(2x^4 - x^3 + x - 1) - (x^4 + x)(8x^3 - 3x^2 + 1)]/$
   $(2x^4 - x^3 + x - 1)^2$

   (c) $-(5x^2 + 4x + 3)/(x^4 + x^3 + x^2)^2$

9. $f'(0)$ exists and equals zero because $\lim_{\Delta x \to 0} \{[f(0 + \Delta x) - f(0)]/\Delta x\}$ is equal to zero independent of whether we let $\Delta x \to 0$ from the left or the right.

10. (a) $y = 5 - 2x$

    (b) $(5/2, 0)$

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